# 6.2 Antidifferentiation by Substitution

**DEFINITION Indefinite Integral**

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of $f$ with respect to $x$** and is denoted by $\int f(x)dx$.

If $F$ is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, where $C$ is an arbitrary constant, called the **constant of integration.**

**EXAMPLE 1 Evaluating an Indefinite Integral**

Evaluate $\int (x^2 - \sin x)\, dx$.

## Properties of Indefinite Integrals

$$\int k\,f(x)dx = k\int f(x)\,dx \quad \text{for any constant } k$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

## Power Formulas

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \text{ when } n \neq -1$$

$$\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$$
(see Example 2)

## Trigonometric Formulas

$$\int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C$$

## Exponential and Logarithmic Formulas

$$\int e^u\,du = e^u + C$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \ln u\,du = u\ln u - u + C \quad \text{(See Example 2)}$$

$$\int \log_a u\,du = \int \frac{\ln u}{\ln a}\,du = \frac{u\ln u - u}{\ln a} + C$$

**EXAMPLE 2 Verifying Antiderivative Formulas**

Verify the antiderivative formulas:

**(a)** $\int u^{-1}\, du = \int \frac{1}{u}\, du = \ln|u| + C$ **(b)** $\int \ln u\, du = u \ln u - u + C$

**EXPLORATION 1** **Are $\int f(u)\,du$ and $\int f(u)\,dx$ the Same Thing?**

Let $u = x^2$ and let $f(u) = u^3$.

1. Find $\int f(u)\,du$ as a function of $u$.
2. Use your answer to question 1 to write $\int f(u)\,du$ as a function of $x$.
3. Show that $f(u) = x^6$ and find $\int f(u)\,dx$ as a function of $x$.
4. Are the answers to questions 2 and 3 the same?

**EXAMPLE 3** **Paying Attention to the Differential**

Let $f(x) = x^3 + 1$ and let $u = x^2$. Find each of the following antiderivatives in terms of $x$:

**(a)** $\displaystyle\int f(x)\,dx$ **(b)** $\displaystyle\int f(u)\,du$ **(c)** $\displaystyle\int f(u)\,dx$

U Substitution Activity

**EXAMPLE 4 Using Substitution**

Evaluate $\int \sin x \, e^{\cos x} \, dx$ .

## EXAMPLE 5 Using Substitution

Evaluate $\int x^2\sqrt{5+2x^3}\,dx$.

**EXAMPLE 6 Using Substitution**

Evaluate $\int \cot 7x \, dx$.

$$\int \frac{\cos 7x}{\sin 7x}$$

**EXAMPLE 7** **Setting Up a Substitution with a Trigonometric Identity**

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a) $\displaystyle\int \frac{dx}{\cos^2 2x}$ (b) $\displaystyle\int \cot^2 3x\,dx$ (c) $\displaystyle\int \cos^3 x\,dx$

a) $\displaystyle\int \sec^2 2x\,dx$

$u = 2x$

$du = 2\,dx$

$dx = \frac{1}{2}du$

$\displaystyle\int \sec^2 u \,\frac{1}{2}\,du$

$= \frac{1}{2}\tan u + C$

$= \frac{1}{2}\tan 2x + C$

b) $\displaystyle\int \cot^2 3x\,dx$

$\displaystyle\int (\csc^2 3x - 1)\,dx$

$u = 3x$

$du = 3\,dx$

$dx = \frac{1}{3}du$

$\displaystyle\int (\csc^2 u - 1)\,\frac{1}{3}\,du$

$\frac{1}{3}(\cot u - u) + C$

$= -\frac{1}{3}\cot 3x - X + C$

c) $\displaystyle\int \cos^3 x\,dx$

$\displaystyle\int \cos^2 x \,\cos x\,dx$

$\displaystyle\int (1-\sin^2 x)\cos x\,dx$

$u = \sin x$

$du = \cos x\,dx$

$dx = \dfrac{du}{\cos x}$

$\displaystyle\int (1-u^2)\,\cancel{\cos x}\,\frac{du}{\cancel{\cos x}}$

$= u - \frac{1}{3}u^3 + C$

$= \sin x - \frac{1}{3}\sin^3 x + C$

**EXAMPLE 8 Evaluating a Definite Integral by Substitution**

Evaluate $\int_0^{\pi/3} \tan x \sec^2 x\, dx$.

$\tan \frac{\pi}{3} = \sqrt{3}$

$\tan 0 = 0$

$u = \tan x$

$du = \sec^2 x\, dx$

$dx = \frac{du}{\sec^2 x}$

$\int_0^{\sqrt{3}} u \cancel{\sec^2 x} \frac{du}{\cancel{\sec^2 x}}$

$\frac{1}{2} u^2 \Big|_0^{\sqrt{3}}$

$= \frac{3}{2}$

**EXAMPLE 9 That Absolute Value Again**

Evaluate $\int_0^1 \frac{x}{x^2 - 4}\,dx.$

$1^2 - 4 = -3$

$0^2 - 4 = -4$

$u = x^2 - 4$

$du = 2x\,dx$

$dx = \frac{1}{2}\frac{du}{x}$

$$\frac{1}{2}\int_{-4}^{-3} \frac{\cancel{x}}{u}\,\frac{du}{\cancel{x}}$$

$$= \frac{1}{2}\left[\ln|u|\right]\Big|_{-4}^{-3}$$

$$= \frac{1}{2}\ln 3 - \frac{1}{2}\ln 4$$

$$= \frac{1}{2}\ln\frac{3}{4}$$